Akaike-type criteria and the reliability of inference: Model selection versus statistical model specification

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ABSTRACT

Since the 1990s, the Akaike Information Criterion (AIC) and its various modifications/extensions, including BIC, have found wide applicability in econometrics as objective procedures that can be used to select parsimonious statistical models. The aim of this paper is to argue that these model selection procedures invariably give rise to unreliable inferences, primarily because their choice within a prespecified family of models (a) assumes away the problem of model validation, and (b) ignores the relevant error probabilities. This paper argues for a return to the original statistical model specification problem, as envisaged by Fisher (1922), where the task is understood as one of selecting a statistical model in such a way as to render the particular data a truly typical realization of the stochastic process specified by the model in question. The key to addressing this problem is to replace trading goodness-of-fit against parsimony with statistical adequacy as the sole criterion for when a fitted model accounts for the regularities in the data.

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1. Introduction

Fisher (1922) pioneered modern frequentist statistics as a model-based approach to statistical induction anchored on the notion of a statistical model, formalized by

\[ M_\theta(z) = \{ f(z; \theta), \theta \in \Theta \}, \quad z \in \mathbb{R}_n^*, \Theta \subset \mathbb{R}^m, m < n. \] (1)

Unlike Pearson (1920), who would proceed from data \( z_0 := (z_1, \ldots, z_n) \) in search of a frequency curve \( f(z; \theta) \) to describe its histogram, Fisher proposed to begin with a prespecified \( M_\theta(z) \) (a ‘hypothetical infinite population’), and view \( z_0 \) as a realization thereof. He envisaged the specification of \( M_\theta(z) \) as a response to the question: ‘Of what population is this a random sample?’ (ibid., p. 313), underscoring that: “the adequacy of our choice may be tested a posteriori.” (p. 314) He identified the ‘problems of statistics’ to be: (1) specification, (2) estimation and (3) distribution and emphasized that addressing (2)-(3) depended crucially on dealing with (1) successfully first. A mis-specified \( M_\theta(z) \) would vitiate any procedure relying on \( f(z; \theta) \), \( z \in \mathbb{R}_n^* \) (or the likelihood function), rendering all inductive inferences unreliable by giving rise to actual error probabilities that are invariably different from the nominal ones.

Despite its manifest importance, specification received only scant attention in the statistics literature since the 1920s:

“The current statistical methodology is mostly model-based, without any specific rules for model selection or validating a specified model.” (Rao, 2004, p. 2)

The initial emphasis was placed on the estimation and distribution facets of statistics. Indeed, the formal apparatus of frequentist statistical inference (estimation, testing and prediction) was largely
in place by the late 1930s. The nature of the underlying *inductive reasoning*, however, was clouded in disagreements. Fisher argued for ‘inductive inference’ spearheaded by his significance testing (Fisher, 1955), and Neyman argued for ‘inductive behavior’ based on Neyman–Pearson (N–P) testing (Neyman, 1956). However, neither account gave a satisfactory answer to the basic question:

> When do data \(z_0\) provide evidence for (or against) a substantive claim \(H\)?

Both testing procedures were plagued by several misconceptions and fallacies, including the *fallacy of acceptance* (no evidence against the null is misinterpreted as evidence for it), and the *fallacy of rejection* (evidence against the null is misinterpreted as evidence for some alternative); see Mayo (1996). Indeed, several crucial foundational problems reverberate largely unresolved to this day:

\[\text{(I)} \quad \text{How could a (possibly) infinite set } \mathcal{P}(z) - \text{ of all possible models that could have given rise to data } z_0 - \text{ be narrowed down to a single statistical model } \mathcal{M}_a(z)?\]

\[\text{(II)} \quad \text{How could the adequacy of a statistical model } \mathcal{M}_a(z) \text{ be tested a posteriori?}\]

\[\text{(III)} \quad \text{What is the role of substantive information in statistical modeling?}\]

\[\text{(IV)} \quad \text{What is the role of pre-data versus post-data error probabilities? (Hacking, 1965).}\]

\[\text{(V)} \quad \text{How could the fallacies of acceptance and rejection be addressed in practice?}\]

These issues created endless confusions in the minds of practitioners concerning the appropriate use and interpretation of the frequentist approach to inference; see Spanos (forthcoming).

Over the last three decades, Fisher’s specification problem has been recast in the form of model selection where questions (IV)–(V) are ignored, but (I)–(III) are dealt with in specific ways. Question (I) is handled by separating the problem into two stages, where a broader family of models \(\mathcal{M}_a(z), i = 1, \ldots, m\) is selected first, and then a best model \(\mathcal{M}_{a*}(z)\) within this family is chosen. The problem raised in (II) is treated by trading goodness-of-fit against parsimony, and the issue in (III) is often handled by using substantive information (including mathematical approximation theory) in selecting the family of models. The quintessential example of such a procedure is based on the *Akaike Information Criterion* (AIC) and variations/extensions thereof, such as the Bayesian (BIC), the Schwarz (SIC), the Hannan–Quinn (HQIC) and the Minimum Description Length (MDL), as well as cross-validation criteria; see Rao and Wu (2001), Burnham and Anderson (2002) and Konishi and Kitagawa (2008).

These Akaike-type procedures are widely used in econometrics, and other applied disciplines, as offering objective methods for selecting parsimonious models (see Greene, 2003). In philosophy of science they are viewed as providing a pertinent way to address the curve-fitting problem; see Forster and Sober (1994) and Kieseppa (1997).

The primary objective of this paper is to make a case that the Akaike-type model selection procedures invariably give rise to unreliable inferences because:

\[\text{(a)} \quad \text{they ignore the preliminary step of validating the presupposed family of models, and}\]

\[\text{(b)} \quad \text{their selection amounts to testing comparisons among the models within the presupposed family but without ‘controlling’ the relevant error probabilities.}\]

To deal with problems (a)–(b) and address questions (I)–(V) the paper argues for a return to the original specification problem, as envisaged by Fisher (1922), where the task is understood as one of selecting a statistical model that renders the data a typical realization thereof, or equivalently, the postulated statistical model ‘accounts for the regularities in the data’. What is different is that the specification problem will be discussed in the context of a refinement/extension of the original F–N–P frequentist framework, known as *error statistics*; see Mayo and Spanos (2010, forthcoming). The refinement, aiming to address problems (I)–(III), focuses on obviating potential errors at the two points of nexus between inductive inference and the real-world phenomenon of interest (Spanos, 2010):

[A] from the phenomenon of interest to an adequate statistical model, and

[B] from inference results to evidence for or against substantive hypotheses or claims.

The extension, aiming to deal with problems (IV)–(V), comes in the form of a *post-data* evaluation of inference based on *severe testing reasoning* (Mayo, 1996).

The key to implementing Fisher’s envisioned specification is provided by distinguishing, *ab initio*, between substantive and statistical information and devising a purely probabilistic construal of a statistical model \(\mathcal{M}_a(z)\). This can be achieved by viewing \(\mathcal{M}_a(z)\) as a parameterization of the stochastic process \(\{z_k, k \in \mathbb{N} : (1, \ldots, n, \ldots)\}\) whose probabilistic structure is chosen so as to render data \(z_0\) a *true typical realization* thereof; see Spanos (1986). The selection of \(\mathcal{M}_a(z)\) in \(\mathcal{P}(z)\) is guided solely by statistical adequacy: the probabilistic assumptions making up the model are valid for data \(z_0\). Securing the statistical adequacy of \(\mathcal{M}_a(z)\) enables one to address problems (a)–(b), and deal with problems (I)–(V) by employing ascertainable error probabilities (pre-data and post-data) to evaluate the reliability and pertinence of inductive inferences, including the appraisal of substantive claims; see Mayo and Spanos (2006).

Section 2 discusses briefly the curve-fitting problem with a view to bringing out the main features of the mathematical approximation perspective that motivates and underlies the Akaike-type selection procedures and most nonparametric methods in econometrics. The error statistical perspective is used to argue that undue reliance on mathematical approximation often underestimates the reliability of inference because it relies on non-testable premises that often ignore the regularities in the data. In Section 3, the Akaike-type selection procedures are discussed and their unreliability illustrated using empirical examples. Section 4 discusses the general question of reconciling statistical and substantive information by embedding a structural model into a validated statistical model. Section 5 contrasts the Akaike-type selection procedures with securing statistical adequacy using thorough misspecification (M-S) testing and respecification, and Section 5 discusses certain charges leveled against M-S testing, like double-use of data, infinite regress, circularity and pre-test bias.

## 2. Mathematical approximation and curve fitting

The curve-fitting problem is often viewed in both statistics and philosophy of science as an exemplar that encapsulates the multitude of dimensions and problems associated with *inductive inference* (learning from data), including *underdetermination* – more than one curve can ‘account for the regularities in the data’ equally well – and the *reliability* of inference, which is often assumed to depend on a priori stipulations, such as the uniformity of nature; see Skyrms (2000).

### 2.1. The curve-fitting problem: A brief summary

In its simplest form, the curve-fitting problem has three basic components:

\[\text{(i)} \quad \text{The existence of a ‘true’ but unknown function: } y = h(x), \quad (x, y) \in \mathbb{R}_x \times \mathbb{R}_y,\]

where \(z_0\) provide evidence for (or against) a substantive claim \(H\).
Appendix

Let

$$\text{(3)}$$

be an attempt to motivate some of mathematical

Powell, 1981)

is largely the result of imposing a

Interestingspecialcasesare

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thereby. Thistheoryproposesspecificanswerstoquestions [a]–[c]

where \( \phi(x), i = 1, \ldots, m \) is a base set of generalized polynomials

defined on \([a, b]\); see the Appendix for a summary of the basic

results on mathematical approximation.

It is important to note that when the approximation is over a net

of points \( \mathcal{G}_m(x) := \{x_k, k = 1, \ldots, n\} \), the results in the Appendix

need to be modified, because the relevant metric is associated with

the discrete \( p \)-norm:

$$\| h(x_k) - g_m(x_k; \alpha) \|_p = \left( \sum_{k=1}^{n} w(x_k) | h(x_k) - g_m(x_k; \alpha) |^p \right)^{\frac{1}{p}},$$

$$p \geq 1,$$

where \( w(x_k) > 0, \sum_{k=1}^{n} w(x_k) = 1 \), denotes the weight function.

This is commonly justified on the basis that the net \( \mathcal{G}_m(x) \) is dense

in the interval \([a, b]\), in the sense that as \( n \to \infty \) the discrete metric

converges to its continuous analogue (see Rivlin, 1981):

$$\lim_{n \to \infty} \left( \sum_{k=1}^{n} w(x_k) | h(x_k) - g_m(x_k; \alpha) |^p \right)^{\frac{1}{p}} = \left( \int_{a}^{b} w(x) | h(x) - g_m(x; \alpha) |^p dx \right)^{\frac{1}{p}}, \quad p \geq 1.$$

Example 1. In an attempt to motivate some of mathematical apparatus needed to address this problem, consider a familiar example where \( g_m(x) \) is chosen to be

$$g_m(x; \alpha) = \alpha_0 + \sum_{k=1}^{m} \alpha_k x^k,$$

i.e. an ordinary polynomial of degree \( m \), and ‘best’ is selected via the

minimization

$$\min_{\alpha \in \mathbb{R}^m} \| \hat{\epsilon}(\alpha) \|^2 = \left( \sum_{k=1}^{n} \left( y_k - \alpha_0 - \sum_{k=1}^{m} \alpha_k x_k^k \right)^2 \right),$$

(3)

giving rise to the least-squares estimator \( \hat{\alpha}_{\text{LS}} = (\hat{\alpha}_0, \ldots, \hat{\alpha}_m) \) of \( \alpha = (\alpha_0, \ldots, \alpha_m) \).

In this example, a number of decisions (choices) were made that

need to be formally (mathematically) justified in light of the

information in (i)–(iii) above:

[a] What guarantees the existence and uniqueness of the ‘best’

fitted curve \( g_m(x; \hat{\alpha}_{\text{LS}}) \)?

[b] Why choose an ordinary polynomial for \( g_m(x; \alpha) \), and not some

other function(s)?

[c] Why minimize the sum of squares in (3), and not some other

objective function?

2.2. The mathematical approximation perspective

Mathematical approximation theory, which began with a

concept of ‘distance’ determining the notion of a ‘best’

approximation.

Mathematically, the most effective way to render the curve

fitting tractable is to view the problem in the context of a normed linear

space (see Powell, 1981), such as the set of all continuous functions

over the interval \([a, b] \subset \mathbb{R} \), denoted by \((C[a, b], \| \cdot \|_p)\),

where the \( p \)-norm is defined by

$$\| f \|_p = \left( \int_{a}^{b} | f(x) |^p dx \right)^{\frac{1}{p}}, \quad p \geq 1, \text{ for all } f \in C[a, b].$$

$$\text{(4)}$$

Interestingly special cases are \( p = 1, 2 \) and \( p = \infty : \| f \|_\infty = \max_{x \in [a, b]} | f(x) | \).

It can be argued that the traditional view of curve fitting

is largely the result of imposing a mathematical approximation

perspective on the problem. Hence, it is no accident that the

approximating function often takes the linearized form

$$\hat{g}_m(x; \alpha) = \hat{\alpha}_0 + \sum_{i=1}^{m} \hat{\alpha}_i \phi_i(x),$$

$$\text{(5)}$$

provide a complete base set for the normed linear (Hilbert) space

(\( L_2(\mathbb{R}), \| \cdot \|_2 \)), and thus, for the approximating function

\( g_m(x_k; \alpha) = \sum_{i=0}^{m} \alpha_i h(x_k) \), the 2-norm ensures both uniqueness

and convergence in mean to \( h(x) \); see Luenberger (1969).

Hence, one can estimate \( \alpha \) using weighted least squares (Hildebrand, 1982):

$$h_0(x) = 1, \quad h_1(x) = 2x, \quad h_2(x) = 4x^2 - 2,$$

$$h_3(x) = x^3 - 3x, \ldots, \quad h_m(x) = (-1)^m e^{x^2} \frac{d^m (e^{-x^2})}{dx^m}.$$

2.3. Mathematical approximation versus inductive inference

2.3.1. Goodness-of-fit versus parsimony: Is that the issue?

The curve-fitting problem discussions are dominated by the

mathematical approximation perspective to such an extent that

the primary problem in selecting (5) is thought to be overfitting,

stemming from the fact that one can make the error in (3) as small

as desired by increasing \( m \). Indeed, the argument goes, one can

make the approximation error \( e(x_k; m) = y_k - \sum_{i=0}^{m} \alpha_i \phi_i(x_k) \) equal

to zero by choosing \( m = n - 1 \); see Skymers (2000). This reasoning

suggests that goodness-of-fit cannot be the sole criterion for ‘best’. To

avoid overfitting, one needs to supplement goodness-of-fit with
pragmatic criteria such as simplicity, which can be justified on prediction grounds, since simpler curves enjoy better predictive accuracy; see Forster and Sober (1994).

A closer look at the above argument reveals that ‘trading goodness-of-fit against parsimony’ ignores two potential unreliability of inference problems when the fitted curve $g_m(x; \hat{\alpha}) = \sum_{k=0}^m \alpha_k x^k$ is used as a basis for inductive inference.

- The first relates to an inherent tension between the sample size $n$ (statistical information) and the degree of approximation $m$ (substantive information) of $g_m(x; \alpha)$ (Section 2.3.2).
- The second has to do with the notion of assessing when ‘a fitted curve accounts for the regularities in data $z_0$’ (Section 2.3.3).

2.3.2. The inherent tension between $m \to \infty$ and $n \to \infty$

The tension between $n$ and $m$ can be brought out by scrutinizing the soundness of the claim that the choice $m = n - 1$ renders the approximation error zero. Although this claim is mathematically sound, it is inherently fallacious because the estimated coefficients $\hat{\alpha} := (\hat{\alpha}_0, \ldots, \hat{\alpha}_m)$ will be inconsistent estimators of $\alpha := (\alpha_0, \ldots, \alpha_m)$, and thus any inference based on $g_m(x; \hat{\alpha})$ will be totally unreliable.

Indeed, this highlights an inherent tension between mathematical convergence results concerning $m \to \infty$, and probabilistic convergence results concerning $n \to \infty$. The former enhances fit by making the approximation error smaller, but consistency (convergence in probability), $\hat{\alpha} \to \alpha$ as $n \to \infty$, is necessary (but not sufficient) for the reliability of any form of inductive inference; see Spanos (2007a).

2.3.3. The qualitative characterization of the approximation error

When does a fitted curve $g_m(x; \hat{\alpha})$ account for the regularities in data $z_0$? Intuition suggests that $g_m(x; \hat{\alpha})$ has captured the systematic information in $z_0$ when what is left, the residuals $\hat{\epsilon}(x; m) = \epsilon(\epsilon(x; m) m \to \infty)$, $k = 1, 2, \ldots, n$, constitute white noise. More formally, when the approximation error $\epsilon(\epsilon(x; m)$ is non-systematic in a probabilistic sense:

\[ [i] \quad \hat{\epsilon}(x; m) \sim \text{IID}(0, \sigma^2), \quad [ii] \quad E[\hat{\epsilon}(x; m) g_m(x; \hat{\alpha})] = 0, \quad \forall (x, k) \in \mathbb{R}^d \times \mathbb{N}, \]

where ‘IID’ stands for ‘independent and identically distributed’. A closer look at the qualitative characterization of $\epsilon(\epsilon(x; m)$ reveals that being ‘small’ in a mathematical sense (see (62) in the Appendix), does not entail (9), and neither does trading goodness-of-fit against overfitting. Indeed, reflecting on the results of mathematical approximation theory suggests that conditions [i]–[ii] will often be invalid since the qualitative characterization of $\epsilon(\epsilon(x; m)$ renders it a function of $k, x$ and $m$. This can potentially devastate the reliability of any inference based on $g_m(x; \hat{\alpha})$.

To be more specific, the potential violation of conditions [i]–[ii] stems from the very mathematical results that specify necessary and sufficient conditions (if) for the existence and uniqueness of the best approximating function $g_m(x; \hat{\alpha})$, known as oscillation-type theorems; see Powell (1981). According to Theorem 3 in the Appendix, the if conditions require the residuals

$\hat{\epsilon}(x; m) = h(x) - g_m(x; \hat{\alpha}), \quad k = 1, \ldots, n$

to alternate in sign at least $m + 2$ times over $G_m(x)$. In practice, $m + 2$ is much too low to render the residuals non-systematic! Analogous oscillation theorems are needed, not only for approximations based on different norms, but also for local (piecewise) polynomial approximation using splines (see Watson, 1980).

This potential unreliability problem can be highlighted using a simple runs test for randomness (IID), which states that under IID the expected number of runs $R$ is approximately $E(R) = (2n - 1)/3$; a run is a subsequence of one type (+ or −) immediately preceded and succeeded by an element of the other type. For $n = 100$, $E(R) \approx 66$, the expected number of runs in the residuals should be considerably higher than $(m + 2)$, otherwise they are likely to exhibit Markov dependence; see Spanos (1999). This problem is illustrated in Spanos (2007a) by comparing the estimated Kepler’s first law of planetary motion (statistically adequate) with Ptolemy’s model (statistically inadequate), shown to represent a quintessential form of curve fitting whose residuals exhibit strong temporal dependence and heterogeneity. Contrary to the widely held belief in philosophy of science, Ptolemy’s model does not account for the regularities in the data!

2.3.4. The missing inductive understructure

Example 3. Consider fitting (by least squares) the line $y = \alpha_0 + \alpha_1 x$, suggested by a certain theory (see Mayo and Spanos, 2004), through the scatter-plot of data $z_0$:

\[ \hat{y}_i = 167.115 + 1.907 x_i, \quad s = 1.77, \quad n = 35. \]

(10)

The end result is that, as it stands, (10) provides no basis for inductive inference (learning from data). At best, mathematical approximation can provide very crude Jackson-type upper bounds (see the Appendix) in terms of $m$ (the degree of $g_m(x; \alpha)$). The residuals $\hat{\epsilon}_i = y_i - \tilde{y}_i$ can be used to construct goodness-of-fit measures like

\[ R^2 = 1 - \frac{\sum_{k=1}^n \hat{\epsilon}_i^2}{\sum_{k=1}^n (y_i - \bar{y})^2}. \]

However, this framework does not provide the necessary understructure for inductive inference, i.e. it does not delimitate the probabilistic premises stating the conditions under which the statistics $(\hat{\alpha}_0, \hat{\alpha}_1, s^2, R^2)$ are inferentially reliable, as opposed to mathematically justifiable or theoretically meaningful.

The above discussion brings out a crucial weakness of the mathematical approximation perspective: its answers to questions [a]–[c] (Section 2.1) rely on premises (a)–(c) (Section 2.2) that are largely non-testable and often ignore the regularities in data $z_0$. This can potentially undermine the reliability of all inductive inference methods that rely on this perspective. Indeed, the non-testability of the invoked mathematical premises can severely undermine the reliability of many nonparametric/semiparametric methods that are misleadingly appealed to as a way to circumvent the statistical miss-specification problem; see Li and Racine (2006).

The question that naturally arises is whether there is way to relate the mathematical premises (a)–(c) (Section 2.2) to the regularities in data $z_0$ that would ensure the validity of assumptions [i]–[ii]. Such a connection is provided by embedding the mathematical approximation problem into a statistical model $M_{\mathcal{Q}}(z)$ whose probabilistic assumptions are chosen with a view to account for the regularities in data $z_0$. This embedding offers a way to transform the non-testable mathematical premises (a)–(c) into the testable [vis-à-vis $z_0$] inductive premises of $M_{\mathcal{Q}}(z)$; see Section 4.

3. Model selection in statistics

3.1. Gauss’s pioneering contribution

Historically, Gauss (1809) should be credited with the first attempt to embed the mathematical approximation formulation into a statistical model $M_{\mathcal{Q}}(z)$, by making explicit probabilistic
assumptions pertaining to a generic error term $\varepsilon_k$, specifying the Gauss linear model (see Spanos, 1986, ch. 18):

$$y_k = \beta_0 + \sum_{i=1}^{m} \beta_i x_{ik} + \varepsilon_k, \quad \varepsilon_k \sim \text{NIID}(0, \sigma^2),$$

$$k = 1, 2, \ldots, n, \ldots$$

(12)

What makes his contribution all-important is that (12) provides the inductive premises for assessing the reliability of inference based on the fitted model $y_k = \hat{\beta}_0 + \sum_{i=1}^{m} \hat{\beta}_i x_{ik}$.

Example 3 (Continued). Gauss’s error assumptions in (12) provide specific premises for inductive inference, in the sense that the estimated model,

$$y_k = 167.115 + 1.907 x_k + \tilde{u}_k,$$

$$R^2 = .995, s = 1.77, n = 35,$$

(13)

can be used to generate inferential statistics, such as the standard errors (SEs) given in parentheses, to draw inferences, including testing the significance of the coefficients ($\beta_0, \beta_1$). For instance, if one were to take these SEs at face value, the $t$-ratios $t(\hat{\beta}_0) = 273.96[0.000], t(\hat{\beta}_1) = 79.458[0.000]$ (p-values in square brackets), would suggest that both coefficients are significantly different from zero. As argued below, however, such inferences turn out to be unreliable when the presumed probabilistic structure in (12) is invalid for data $x_0 = (y_1, \ldots, y_m; x_1, \ldots, x_n)$.

In general, the appended probabilistic structure raises two crucial questions:

(A) How does the approximation error $\varepsilon(x; m)$ relate to the statistical error $\varepsilon_k$?

(B) How does one validate the probabilistic assumptions of the error $\varepsilon_k$?

These questions are crucial when reconciling substantive and statistical information with a view to ensure the reliability of inductive inference; see Section 4.

To illustrate the built-in connection between the mathematical (a)–(c) (Section 2.2) and inductive premises, consider the choice of a norm $\|\cdot\|$ and the distributional assumption underlying a statistical model $\mathcal{M}_n(\mathbf{z})$.

3.2. The choice of a norm and distributional assumptions

The majority of statistical models motivated by the mathematical approximation perspective revolve around a generalized form of (12):

$$y_k = \alpha_0 + \sum_{i=1}^{m} \alpha_i \phi(x_k) + \varepsilon_k, \quad \varepsilon_k \sim \text{NIID}(0, \sigma^2),$$

$$k = 1, 2, \ldots, n, \ldots$$

(14)

where $g_m(x; \alpha) = \alpha_0 + \sum_{i=1}^{m} \alpha_i \phi(x_k)$ is chosen using substantive information relating to $y = h(x), (x, y) \in [\mathcal{R}_X \times \mathcal{R}_Y]$. In this case, the assumed distribution is Normal and the norm implicitly used is $\|\cdot\|_2$, inducing the inner product,

$$(h(x_k) - g_m(x; \alpha))_2 = \sum_{k=1}^{n} \left( y_k - \alpha_0 - \sum_{i=1}^{m} \alpha_i \phi(x_k) \right)^2,$$

(15)

and convergence in mean, giving rise to the least-squares estimator $\hat{\alpha}_L$ of $\alpha$. Indeed, it is no accident that in Example 2 a natural base set for $L_2(-\infty, \infty)$ is provided by the Hermite polynomials whose weight function for orthogonality $w(x) = e^{-x^2}$ is in essence [by rescaling] the standard Normal density $\phi(x)$; i.e.

the Rodrigues formula for $h_m(x)$ can be written equivalently as $h_m(x) = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} \frac{d^m}{dx^m} \phi(x), \ m = 0, 1, \ldots$. What is often not made explicit in the statistics literature is the connection between other $p$-norms and the implicit distributional assumption. The 1-norm $\|h(x_k) - g_m(x; \alpha)\|_1$, inducing the inner product (Powell, 1981),

$$h(x_k) - g_m(x; \alpha) = \sum_{k=1}^{n} \left( y_k - \alpha_0 - \sum_{i=1}^{m} \alpha_i \phi(x_k) \right),$$

(16)

and pointwise convergence, giving rise to the least absolute deviation estimator of $\alpha$ (Shao, 2003), are related to the Laplace distribution, $f(\varepsilon_k) = (\frac{\sigma}{2})e^{-|\varepsilon_k|/\sigma}, \varepsilon_k \in \mathbb{R}$.

The 1-norm $\|h(x_k) - g_m(x; \alpha)\|_\infty$, inducing the inner product,

$$\sup_{\varepsilon_k \in [-\sigma, \sigma]} \|y_k - \alpha_0 - \sum_{i=1}^{m} \alpha_i \phi(x_k)\|,$$

(17)

and uniform convergence, giving rise to the Minimax estimator of $\alpha$ (see Shao, 2003), are related to the uniform distribution, $f(\varepsilon_k) = \frac{1}{2\sigma}, \varepsilon_k \in [-\sigma, \sigma]$. Note that the natural base set for $[1, -1]$ is provided by the Legendre polynomials whose weight function $w(x)$ for orthogonality relates to the uniform density; see Hildebrand (1982).

The importance of this connection stems from the fact that when the choice of a norm is based on the appropriateness of the corresponding distributional assumption vis-à-vis data $x_0$, the resulting estimator $\hat{\alpha}_{L_2}$ is the Maximum Likelihood Estimator of $\alpha$, which enjoys certain desirable statistical properties, including parameterization invariance, which is particularly crucial in econometrics since the structural parameters are often reparameterizations/restrictions of the statistical parameters; see Section 4.4. This suggests that in practice the choice of a norm should not be a matter of convenience, mathematical expediency or even robustness (see Shao, 2003, p. 346), but needs to be justified on statistical adequacy grounds.

3.3. Akaike-type model selection procedures

Akaike-type model selection procedures assume a prespecified family of models $\{\mathcal{M}_n(\mathbf{z}), i = 1, 2, \ldots m\}$ based on some variation/extension of (14). The stated objective of these procedures is motivated by the curve-fitting perspective and the selection is guided by trading goodness-of-fit against overfitting.

The initial attempt to curb overfitting was made by introducing the adjusted $R^2$,

$$R^2_{m} = 1 - \left( \frac{n-1}{n-m-1} \right)(1-R^2),$$

(18)

that ‘penalizes’ the $R^2$ in (11) for increasing as $m$ increases. The statistical literature continued with Akaike’s (1970) Final Prediction Error (FPE), Mallows’s (1973) $C_p$ and Allen’s (1971) cross-validation criteria:

$$\text{FTE}_N = \frac{\text{RSS}_m}{n} \left( 1 + \frac{2K}{n-K} \right),$$

$$C_p = \frac{(n\text{RSS}_p - n + 2p)}{\text{RSS}_m},$$

$$\text{CV}(1) \simeq \frac{n\text{RSS}_m}{(n-K)^2},$$

(19)

where $K = (m + 1), \text{RSS}_p = \sum_{k=1}^{n} (y_k - \hat{\beta}_0 - \sum_{i=1}^{p} \hat{\beta}_i x_{ik})^2$ for $p \leq m$.

The model selection literature came of age with Akaike’s Information Criterion (AIC) and related procedures such as the BIC,
SIC, HQIC in the 1970s and 1980s; see Rao and Wu (2001). The AIC is based on curbing overfitting by penalizing the log-likelihood function \( \ln(L(\theta)) \) – measuring goodness-of-fit – using the number of unknown parameters \( K \) in \( \theta \):

\[
AIC = -2 \ln L(\theta) + 2K.
\]

The formal justification for this criterion is that AIC provides a pertinent estimator of the risk of the Kullback–Leibler (K–L) loss function:

\[
\Delta_{K-L}(M_0, \hat{M}_1) = E_0 \left( \frac{f_0(x; \theta_0)}{f_1(x; \hat{\theta})} \right) = \int_{\mathbb{R}^d} \left( \ln f_0(z; \theta_0) / f_1(z; \hat{\theta}) \right) f_1(z; \hat{\theta}) dz,
\]

viewed as the expected discrepancy between the true model \( M_0 \) and an estimated model \( M_1 \). In particular, Akaike (1973) showed that when one compares different models within a prespecified family \( \{M_i(z), i = 1, 2, \ldots, m\} \),

\[
AIC(i) = -2 \ln f_i(z; \hat{\theta}_i) + 2K_i, \quad i = 1, 2, \ldots, m,
\]

provides an asymptotically unbiased estimator of \( \Delta_{K-L}(M_0, \hat{M}_i) \), \( K_i \) being the number of parameters in \( \theta_i \). For example, in the case of (12),

\[
AIC = n \ln(\hat{\sigma}^2) + 2K, \quad \text{or} \quad \text{AIC}_n = n \ln(\hat{\sigma}^2) + \frac{2K}{n},
\]

where \( \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{\theta}_0) - \sum_{i=1}^{m} \hat{\theta}_ik_i)^2 \), and \( K = m + 2 \). The AIC procedure selects \( M_0(z) \) when \( AIC(k) \) yields the minimum value. Early studies on the effectiveness of the AIC raised two crucial weaknesses (see McQuarrie and Tsai, 1989):

(i) The value \( \hat{K} \) chosen by AIC is an inconsistent estimator of the ‘true’ \( K^* \).

(ii) In small samples, AIC often leads to overfitting, i.e. \( \hat{K} > K^* \).

Attempts to deal with the inconsistency problem gave rise to several modifications of the AIC, with the most widely used being the BIC (see Schwarz, 1978), Hannan and Quinn (1979) and Rissanen (1978) Minimum Description Length criteria:

\[
\text{BIC}_n = n \ln(\hat{\sigma}^2) - \frac{K \ln(n)}{n},
\]

\[
\text{HQIC}_n = n \ln(\hat{\sigma}^2) - \frac{2K \ln(n)}{n},
\]

\[
\text{MDL}_n = (\text{BIC}_k / 2).
\]

These are special cases of the generalized criterion GIC\( _n = -2 \ln f(Z; \hat{\theta}) + a(n) \cdot K \), where \( a(n) \) is a smooth function of the sample size, which also includes Tachcheu’s (TIC) criterion; see Konishi and Kitagawa (2008).

Attempts to deal with the small sample overfitting problem led to a several modifications of AIC, such as (see McQuarrie and Tsai, 1998)

\[
\text{M-AIC}_n = n \ln(\hat{\sigma}^2) + (\ln(K) / \ln(n) - K) \cdot 2).
\]

In what follows, it is argued that (i)–(ii) above are the least of AIC’s difficulties. The unreliability of inference problems (a)–(b) (Section 1) are considerably more serious, and undermine equally all the modifications/extensions of the AIC, including cross-validation criteria and nonparametric methods which rely solely on mathematical approximation theory (see Konishi and Kitagawa (2008). In all these areas, as well as curve fitting (see Forster and Sober, 1994), the assumptions defining the likelihood function are often treated as an afterthought, and not as crucial stipulations whose inappropriateness vis-à-vis data \( z_0 \) will undermine the reliability of inference.

### 3.4. Potential errors in model selection

Viewing the Akaike-type model selection procedures in the context of error statistics (Mayo and Spanos, forthcoming) calls for probing all the different ways the final inference \( M_{q_0}(z) \) is the ‘best’ model within the prespecified family \( \{M_i(z), i = 1, 2, \ldots, m\} \) might be in error, as well as delineating the notion of a ‘best’ model as it relates to the various objectives associated with using \( M_{q_0}(z) \).

It is argued that the primary objective of a statistical model \( M_{q_0}(z) \) is to provide a sound basis for both inductive inference (estimation, testing, prediction and simulation) as well as risk evaluation associated with different decision rules. In light of that, a minimum requirement for a ‘best’ model \( M_{q_0}(z) \) is statistical adequacy, because without it all inductive inferences, as well as risk evaluations, are likely to be misleadingly unreliable. This stems from the fact that a mis-specified \( f(z; \theta) \) will vitiate the sampling distribution \( F_n(t) \) of any statistic \( T_n = g(Z_1, \ldots, Z_n) \), since, by definition,

\[
F_n(t) = P(T_n \leq t) = \int_{\mathbb{R}} \int_{\{z: f(z; \theta)\}} f_{(z; \theta)} dz, \quad t \in \mathbb{R}.
\]

In turn, an invalid \( F_n(t) \) will undermine the reliability of any inference based on it by giving rise to a systematic divergence between actual and nominal error probabilities.

**Example 4.** Let \( M_{q_0}(z) \) be a simple Normal model, i.e. \( Z_k \sim \text{NID}(\mu, \sigma^2), k = 1, \ldots, n \), and consider testing \( H_0 : \mu = \mu_0 \) versus \( H_1 : \mu > \mu_0 \), using the \( t \)-test:

\[
\tau(Z) = \frac{\sqrt{n} (Z_n - \mu_0)}{s}, \quad c_1 := \{ Z : \tau(Z) > c_\alpha \},
\]

\[
\bar{Z}_n = \frac{1}{n} \sum_{k=1}^{n} Z_k, \quad s^2 = \left( \frac{1}{n-1} \right) \sum_{k=1}^{n} (Z_k - \bar{Z}_n)^2.
\]

What would be the effect on the relevant error probabilities be if Independence is false? Instead, assume that

\[
\text{Corr}(Z_i, Z_j) = \rho, \quad 0 < \rho < 1, \quad \text{for } i \neq j, i, j = 1, \ldots, n.
\]

For \( n = 100 \), and a nominal \( \alpha = .05 \) (\( c_\alpha = 2.01 \)), even a small value \( \rho = .1 \) will yield an actual type I error of \( \alpha^* = .317 \), a sixfold increase! Similarly, the distortions in power, ranging from positive for tiny discrepancies \( \gamma = \mu - \mu_0, \pi^*(\gamma = .01) - \pi(\gamma = .01) = .266 \) [where more power is not needed], to negative for larger discrepancies, \( \pi^*(\gamma = .3) - \pi(\gamma = .3) = -.291 \) [where more power is needed], will render the test completely unreliable. Moreover, the distortions in both the type I error and the power get much larger as \( \rho \rightarrow 1 \); see Spanos (2009).

Similarly, the evaluation of the risk function for a decision rule \( \hat{\theta}(Z) \) using a loss function \( L(\theta) \) is likely to be highly misleading because the averaging is with respect to a mis-specified \( f(z; \theta) \) vitiating the result:

\[
\mathcal{R} \{ \hat{\theta}(Z, \theta) = \int_{\mathbb{R}} \int_{\{z: f(z; \theta)\}} L(\hat{\theta}(z, \theta) \cdot f(z; \theta) dz, \quad \theta \in \Theta.
\]

In the Akaike-type model selection literature the notion of ‘best’ is sometimes identified with the ‘true’ model, whatever that might mean (see Burnham and Anderson, 2002). In the error statistical framework that would require one to secure both statistical and substantive adequacy – \( M_{q_0}(z) \) provides a veritable explanation for the phenomenon of interest – which calls for additional probing of (potential) errors in bridging the gap between the two. In
the present context, the notion of a ‘best’ model is assumed to include statistical adequacy [accounting for the chance regularities in data $z_0$], since it is necessary for assessing substantive adequacy.

The first potential error arises when the prespecified family \( \{M_q(z), i = 1, \ldots, m\} \) does not include an adequate model $M_q(z)$. This is a variant of the statistical adequacy problem that arises when Akaike-type model selection procedures take the likelihood function at face value and ignore the problem of model validation. This will invariably lead astray inferring a best model because any use of error probabilities (or risks) will be misleading; see Section 3.5.

A second error arises when the family \( \{M_q(z), i = 1, \ldots, m\} \) does include a statistically adequate model, say $M_q(z)$, but is different from the selected model $M_q(z), j \neq k$, which is inadequate. Model selection procedures ignore this potential error because, despite edicts to the contrary (see Burnham and Anderson, 2002), their norm-based minimization is tantamount to testing comparisons among the models within the prespecified family using Neyman–Pearson (N–P) testing, but without ‘controlling’ the relevant error (type I and II) probabilities; see Section 3.6.

### 3.5. Selection within a mis-specified family of models

**Example 3 (Continued).** Consider using the Normal/linear regression model (Table 2) to estimate the relationship between $y_t$, the population of the USA and $x_t$—a special predictor, using annual data for 1955–1989 (see Mayo and Spanos, 2004):

$$M(1) : y_t = 167.115 + 1.907x_t + \hat{u}_t,$$

$$R^2 = .995, s = 1.77, n = 35. \quad \text{(28)}$$

Despite the high $R^2$, consider exploring the possibility of using Akaike-type procedures to choose $m = K - 1$ within the broader family of models

$$M(m) : y_t = \alpha_0 + \sum_{i=1}^{m} \alpha_i \psi_i(x_t) + \varepsilon_t,$$

where $\{\psi_i(x_t), i = 1, \ldots, m\}$ are orthogonal Chebyshev polynomials; see Powell (1981).

The results in Table 1 indicate that all three criteria AIC, BIC and HQIC, as well as the modified AIC in (25), select the same model: $M(4)$.

Applying certain simple mis-specification (M-S) tests (see Spanos and McGuirk, 2001), it can be shown (Table 3) that (28) is statistically mis-specified (assumptions [4]–[5] are invalid); $p$-values are reported in square brackets. Because of the statistical inadequacy of (29), the Akaike-type model selection procedures will always make fallacious selections; this issue was first highlighted by Lehmann (1990, p. 162). Note that this criticism extends to all the above selection procedures (see Rao and Wu, 2001). Statistical mis-specification will give rise to unreliable inferences concerning the value of $m$ with probability one.

### 3.6. Selection within a well-specified family of models

Although the scenario that the prespecified family of models is well-specified is highly unlikely in practice, it is interesting to demonstrate that even in this best-case scenario model selection procedures are likely give rise to misleading inferences.

**Example 3 (Continued).** Using statistical adequacy as the criterion for ‘best’, one is led to the family of statistical models (see Mayo and Spanos, 2004):

$$M(k, \ell) : y_t = \beta_0 + \beta_1 x_t + \sum_{i=1}^{k} \delta_i t + \sum_{i=1}^{\ell} \alpha_i y_{t-i} + \gamma x_{t-i} + \varepsilon_t,$$

$$t \in \mathbb{N}, \quad \text{(30)}$$

which is the Dynamic Linear Regression (DLR) model with trends; see Spanos (1986). The question is whether Akaike-type criteria will choose the correct model, knowing that an adequate model (it accounts for the chance regularities in the data) exists.

The results in Table 4 indicate that the model selected by the AIC and HQIC criteria is $M(3, 2)$, and $M(1, 2)$ is selected by the BIC. It turns out that the model selected on statistical adequacy grounds is $M(1, 2)$:

$$y_t = 17.687 + .193 t - .000 x_t + 1.496 y_{t-1} + .013 x_{t-1}$$

$$- .596 y_{t-2} + .014 x_{t-2} + \hat{e}_t,$$

$$R^2 = .9999, s = .154, n = 35. \quad \text{(31)}$$

The statistical adequacy of (31) is established via thorough mis-specification (M-S) testing where all the assumptions underlying the DLR model are validated vis-à-vis the data in question; see Mayo and Spanos (2004).

Parenthetically, the N–P test for the joint significance of the coefficients of $(x_t, x_{t-1}, x_{t-2})$ in (31) yields $F(3, 26) = .142[.934]$, which indicates that the special predictor is totally unrelated to the US population, negating the earlier (unreliable) inference that $x_t$ is an excellent predictor, based on the mis-specified model (28).

The questions that naturally arise are: ‘What led to the different choices?’, and ‘Why did the BIC select the correct model?’ The

### Table 1

<table>
<thead>
<tr>
<th>Model</th>
<th>AIC$_m = \ln(\hat{\sigma}^2) + [2K/n]_m</th>
<th>Rank</th>
<th>BIC$_m$</th>
<th>Rank</th>
<th>HQIC$_m$</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>M(1)</td>
<td>ln(2.9586) + [2(3)/35] = 1.256</td>
<td>5</td>
<td>1.389</td>
<td>4</td>
<td>1.302</td>
<td>4</td>
</tr>
<tr>
<td>M(2)</td>
<td>ln(2.5862) + [2(4)/35] = 1.179</td>
<td>3</td>
<td>1.357</td>
<td>3</td>
<td>1.240</td>
<td>3</td>
</tr>
<tr>
<td>M(3)</td>
<td>ln(2.5862) + [2(5)/35] = 1.236</td>
<td>4</td>
<td>1.458</td>
<td>5</td>
<td>1.313</td>
<td>5</td>
</tr>
<tr>
<td>M(4)</td>
<td>ln(1.8658) + [2(6)/35] = 0.967</td>
<td>1</td>
<td>1.233</td>
<td>1</td>
<td>1.059</td>
<td>1</td>
</tr>
<tr>
<td>M(5)</td>
<td>ln(1.8018) + [2(7)/35] = 0.989</td>
<td>2</td>
<td>1.300</td>
<td>2</td>
<td>1.096</td>
<td>2</td>
</tr>
</tbody>
</table>

### Table 2

| Statistical GM: $y_t = \beta_0 + \beta_1 x_t + \ldots + \beta_k x_{t-k} + \varepsilon_t, t \in \mathbb{N} | $ | $ t \in \mathbb{N} $ |
|-----------------|-----------------|
| [1] Normality:  | $(y_t | x_t = x) \sim N(\cdot, \cdot)$. |
| [2] Linearity:   | $E(y_t | x_t = x) = \beta_0 + \beta_1 x_t$. |
| [3] Homoskedasticity: | $\text{Var}(y_t | x_t = x) = \sigma^2$. |
| [4] Independence: | $(y_t | x_t = x_t, t \in \mathbb{N})$ indep, process, |
| [5] t-invariance: | $(\beta_0, \beta_1, \sigma^2)$ are not changing with $t$. |

### Table 3

<table>
<thead>
<tr>
<th>M-S (M-S) Testing Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normality: $D' AP(s) = -.482[.314]$.</td>
</tr>
<tr>
<td>Linearity: $F(1, 29) = 4.668[.049]$.</td>
</tr>
<tr>
<td>Homoskedasticity: $F(2, 30) = .802[.458]$.</td>
</tr>
<tr>
<td>Independence: $F(1, 31) = 6.608[.015]^*$.</td>
</tr>
<tr>
<td>t-invariance: $F(1, 29) = 156.273[.000]^* $.</td>
</tr>
</tbody>
</table>
The main conclusion one can draw from the above empirical examples is that the Akaike-type procedures will only circumspectly select a statistically adequate model and always unknown to the modeler. The next section summarizes the error statistical modeling framework where statistical adequacy can be deliberately secured with a view to ensure the reliable appraisal of substantive information.

4. Bridging over statistical and structural models

In the error statistical framework the reconciliation between the statistical and substantive information is viewed in the broader context of bridging the gap between theory and data z₀ using a sequence of interconnecting models: theory, structural (estimable) and statistical models; see Spanos (1986).

4.1. Statistical versus substantive information

It is generally acknowledged that both substantive and statistical information play important roles in learning from data, and in practice empirical models constitute an amalgam of both sources of information, but the role of each type of information has not been delineated in modern statistics; Lehmann (1990) and Spanos (2006a).

In the curve-fitting problem, the family of models is determined mainly by approximation theory (substantive) information based on the ‘smoothness’ of y = h(x), (x, y) ∈ ℝₓ × ℝᵧ. For instance, in Example 2, this theory asserts that when h(x) ∈ L₂(−∞, ∞), the Hermite orthogonal polynomials provide a complete base set, and the 2-norm ensures the existence and uniqueness of gₓ(x; α) = ∑₁₀₀₀αₖhₖ(x). None of these choices, however, has anything to do with the chance regularities in data z₀.

Broadly speaking, statistical information refers to the chance regularity patterns exhibited by the data when viewed as realizations of generic stochastic processes, without any information pertaining to what they represent (measure) substantively. For instance, in Fig. 1 one can see a typical realization of a process yₖ, k ∈ ℕ assumed to be IID, but Fig. 2 depicts a typical realization of a process Yₖ, k ∈ ℕ assumed to be Normal, Markov and Stationary. This can be discerned from these plots using analogical reasoning; see Spanos (1999, ch. 5).

4.2. From a theory to a structural model

The term theory is used generically as any claim conjectured to elucidate a phenomenon of interest. When one proposes a theory to explain the behavior of an observable variable, say yₓ, one demarcates the segment of reality to be modeled by selecting the primary influencing factors xₛ, aware that there might be numerous other potentially relevant factors xₐ (observable and unobservable) influencing the behavior of yₓ.
The assumptions [i]–[iii] are clearly non-testable vis-à-vis data \( \mathbf{z}_0 \) because their confirmation would involve all possible values of both \( \mathbf{x}_k \) and \( \xi_k \). To render them testable vis-à-vis data \( \mathbf{z}_0 \) one needs to embed the structural model (36) into a statistical model built on chance regularities in \( \mathbf{z}_0 \) reflecting the probabilistic structure of \( \{ \mathbf{z}_k \} \), \( k \in \mathbb{N} \); a crucial modeling move that also addresses questions (A)–(B) of Section 3.1. The embedding depends crucially on the nature of the available data \( \mathbf{z}_0 \) and their relation to the theory in question; sometimes the gap between them might be unbridgeable (see Spanos, 1995).

### 4.3. Embedding a structural into a statistical model

The nature of the embedding itself depends on whether the data \( \mathbf{z}_0 \) are the result of an experiment or they are observational in nature, but the aim in both cases is to find a way to transform the substantive error \( \epsilon(\mathbf{x}_k, \xi_k) \), for all \( (\mathbf{x}_k, \xi_k) \in \mathbb{R}_x \times \mathbb{R}_k \), into a generic IID process without the quantifier; see Spanos (2006a).

**Experimental data.** In the case where one can perform experiments, controls and ‘experimental design’ techniques such as replication, randomization and blocking, can often be used to ‘neutralize’ and ‘isolate’ the phenomenon from the potential effects of \( \xi_k \) by ensuring that the uncontrolled factors cancel each other out; see Fisher (1935). The objective is to transform (\( \sim \)) \( \epsilon(\mathbf{x}_k, \xi_k) \) into a generic IID error:

\[
\sim \epsilon_k \sim \text{IID}(0, \sigma^2), \quad k = 1, \ldots, n.
\]

This, in effect, embeds the structural model (36) into a statistical model \( \mathcal{M}_\theta(\mathbf{z}) \):

\[
y_k = h(\mathbf{x}_k; \varphi) + \epsilon_k, \quad \epsilon_k \sim \text{IID}(0, \sigma^2), \quad k = 1, 2, \ldots, n,
\]

where the statistical error term \( \epsilon_k \) in (41) is qualitatively very different from the substantive error term \( \epsilon(\mathbf{x}_k, \xi_k) \) in (36), because \( \epsilon_k \) is no longer a function of \( (\mathbf{x}_k, \xi_k) \), and its assumptions are rendered testable vis-à-vis data \( \mathbf{z}_0 \); see Spanos (2006a). A widely used special case of (41) is the Gaussian linear model (12).

**Observational data.** In the case of observational (non-experimental) data \( \mathbf{z}_0 \), the embedding takes a different form in the sense that the experimental control and intervention are replaced by judicious conditioning on an appropriate information set \( \mathcal{D}_0 \), often generated by an observable process, say \( \sigma(\mathbf{X}_k) \). The generating mechanism of the embedding statistical model takes the general form

\[
y_k = E(\mathbf{y}_k | \mathcal{D}_k) + u_k, \quad k \in \mathbb{N},
\]

where \( \mathbf{y}_k = E(\mathbf{y}_k | \mathcal{D}_k) \) denotes the systematic component and \( \mathcal{D}_k \) is a proper subset of the \( \sigma \)-field \( \mathcal{F} \) in the probability space \( (\mathcal{S}, \mathcal{F}, \mathbb{P}(\cdot)) \), on which the process \( \{ \mathbf{z}_k \} \) is defined. \( \mathcal{D}_k \) is chosen in such a way so as to render the error process \( \{ u_k, \mathcal{D}_k \} \), \( k \in \mathbb{N} \),

\[
u_k = y_k - E(\mathbf{y}_k | \mathcal{D}_k),
\]

non-systematic in the sense of being a Martingale-difference \( (M-d) \) process relative to \( \mathcal{D}_k \); see Doob (1953). The statistical error term is not treated as an autonomous but as a derived process whose probabilistic structure is determined by that of the observable stochastic process \( \{ \mathbf{z}_k, k \in \mathbb{N} \} \).

In specifying a statistical model \( \mathcal{M}_\theta(\mathbf{z}) \) one has a twofold objective in mind:
[I] to account for the chance regularities in data \( z_0 \) by choosing a probabilistic structure for the stochastic process \( \{Z_k, k \in \mathbb{N}\} \) so as to render the data \( z_0 \) a truly typical realizations thereof, and

[II] to parameterize the probabilistic structure of \( \{Z_k, k \in \mathbb{N}\} \) in an attempt to specify an adequate statistical model \( \mathcal{M}_0(z) \) that would embed (nest) the structural model of interest \( \mathcal{M}_y(z) \) in its context.

It is important to emphasize that the functional form of the systematic component \( E(y_k \mid \mathcal{X}_k) \) will be determined exclusively by the statistical information described by the joint \( D(Z_1, Z_2, \ldots, Z_n, \phi) \). For instance, when \( \{Z_k, k \in \mathbb{N}\} \) is IID and \( \mathcal{X}_k = \sigma(X_k) \), the functional form of the regression and stochastic functions, \( E(y_k \mid X_k) = h(X_k) \) and \( \text{Var}(y_k \mid X_k) = g(X_k) \), are determined by \( D(y_k, X_k; \psi) \) since

\[
D(y_k \mid X_k; \theta) = \left[ D(y_k, X_k; \psi) \int_{\psi \in \mathbb{R}} D(y_k, X_k; \psi) \, dy_k \right],
\]

\( \forall(y_k, X_k) \in \mathbb{R}_y \times \mathbb{R}_X \).

When \( D(y_k, X_k; \psi) \) is assumed to be Normal, \( E(y_k \mid X_k) = \beta_0 + \beta_1 X_k \) and \( \text{Var}(y_k \mid X_k) = \sigma^2 \), giving rise to the Normal/linear regression (N/LR) model (Table 2). Indeed, the N/LR model can be formally viewed as a tripartite reduction of the form

\[
D(Z_1, Z_2, \ldots, Z_n; \phi) \overset{\text{NID}}{\longrightarrow} \prod_{k=1}^{n} D(y_k \mid X_k; \theta).
\]

This gives rise to the N/LR model, specified in terms of assumptions [1]–[5] (Table 2). The relationship between the observable process \( \{y_k \mid X_k = x_k\} \in \mathbb{N} \) and the statistical error process, as defined in (43), is

\[
(y_k \mid X_k = x_k) \sim \mathcal{N}(\beta_0 + \beta_1 x_k, \sigma^2) \Rightarrow (y_k \mid X_k = x_k) \sim \text{NID}(0, \sigma^2).
\]

In view of the relationship between the probabilistic structure of the process \( \{Z_k, k \in \mathbb{N}\} \) and the model assumptions, one can use the chance regularities exhibited by data \( z_0 \) to guide the selection of an appropriate statistical model. For example, if the scatter plot of \( z_0 \) looks like Fig. 3, the N/LR model would be an appropriate choice, but if it looks like Fig. 4, the N/LR model will be inappropriate, irrespective of the structural model. This is because Fig. 4 exhibits a typical realization of an exponential IID process whose regression and stochastic functions are (Spanos, 1999)

\[
E(y_k \mid X_k) = \frac{(1 + \theta + x_k) x_k}{(1 + \theta) x_k^2}, \quad \text{Var}(y_k \mid X_k) = \frac{[(1 + \theta + x_k)^2 - 2\theta^2]}{(1 + \theta x_k)^2}, \quad \forall x_k \in \mathbb{R}_+, \theta > 0.
\]

In this sense the statistical model \( \mathcal{M}_0(z) \) is built primarily on statistical information, and has 'a life of its own' in the sense that it constitutes a parameterization of a stochastic process \( \{Z_k, k \in \mathbb{N}\} \), underlyng data \( z_0 \), chosen to account for the chance regularities in data \( z_0 \), which can be gleaned in Figs. 1–4. In this sense, a statistically adequate model \( \mathcal{M}_0(z) \) provides a form of statistical knowledge, against which the substantive information can be appraised. Hence, substantive information enhances learning from data when it does not contravene statistical knowledge.

4.4. Reconciling substantive and statistical information

Substantive subject matter information is crucially important in learning from data about phenomena of interest, but no systematic learning can take place in the context of a statistically misspecified model. For reliable assessment of substantive questions of interest it is imperative to have a statistically adequate model \( \mathcal{M}_0(z) \) which is built on separate information, and therefore can be used to provide the broader inductive premises for evaluating substantive adequacy. The latter requires the structural model \( \mathcal{M}_0(z) \) to provide a veritable explanation for the phenomenon of interest, and necessitates probing (potential) errors in bridging the gap between the two; this includes external validity, confounding effects and other concerns (see Spanos, 2006b,c).

The first step in assessing substantive information is to embed the structural \( \mathcal{M}_y(z) \) into a statistical model \( \mathcal{M}_0(z) \) via reparameterization/restriction, in the form of the implicit function \( \mathcal{G}(\varphi, \theta) = 0 \), where \( \varphi \) and \( \theta \) denote the structural and statistical parameters, respectively. This provides a link between \( \mathcal{M}_0(z) \) and the phenomenon of interest that takes the form of identification:

Does the implicit function \( \mathcal{G}(\varphi, \theta) = 0 \)

define \( \varphi \) uniquely in terms of \( \theta \)?

Often, there are more statistical than structural parameters, and that enables one to test the additional substantive information using the overidentifying restrictions:

\[
\mathcal{H}_0 : \mathcal{G}(\varphi, \theta) = 0, \quad \text{versus} \quad \mathcal{H}_1 : \mathcal{G}(\varphi, \theta) \neq 0.
\]

This error statistical view of identification differs from the traditional textbook notion (see Greene, 2003) in so far as it requires that the underlying statistical model (the reduced form) be validated vis-à-vis data \( z_0 \) for the link between structural parameters and the phenomenon of interest to be rendered trustworthy; see Spanos (1990).

5. Model specification versus model selection

5.1. Statistical model specification

Statistical information refers to the chance regularities (recurring patterns) exhibited by data \( y_0 \) when viewed as a realization of a generic stochastic process \( \{Y_k, k \in \mathbb{N}\} \), irrespective of what the
data quantify (substantively). This enables one to provide a purely probabilistic construal of a statistical model $M_0(y)$, by viewing it as a parameterization of the probabilistic structure of the process $\{Y_t, k \in \mathbb{N}\}$, as summarized by $f(y, \theta), M_0(y)$ is viewed as an idealized probabilistic description of the stochastic mechanism that gave rise to data $y_0$. In this sense, $M_0(y)$ is built exclusively on statistical systematic information in data $y_0$, and its parameterization is chosen so as to embed the structural model $M_0(y)$ in its context; see Section 4.

**Example 5.** Specification begins with a data set $y_0 := (y_{-p}, y_{-p+1}, \ldots, y_1, \ldots, y_p)$, say Fig. 2, and poses the question: ‘What kind of probabilistic structure for $\{Y_t, t \in \mathbb{N}\}$ would render that data a typical realization thereof?’ Using analogical reasoning, one can conjecture that $\{Y_t, t \in \mathbb{N}\}$ being Normal (N), Markov(p) (M) and Stationary (S), would do just that. Imposing these assumptions yields the following reduction:

$$f(Y_1, Y_2, \ldots, Y_n; \theta) \equiv f_0(Y_1, Y_2, \ldots, Y_p; \theta)$$

$$\prod_{t=1}^{n} f(y_t | y_{t-1}, \ldots, y_{t-p}; \theta),$$

where the Normality of $\{Y_t, t \in \mathbb{N}\}$ implies that for $Y_{t-1}^p := (Y_{t-1}, \ldots, Y_{t-p})$:

$$(Y_t, Y_{t-1}^p) \sim N\left(\alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i}, \sigma_0^2\right), \quad t \in \mathbb{N}.$$  

This reduction gives rise to the AR(p) model in terms of complete and internally consistent set of testable [vis-à-vis data $y_0$] probabilistic assumptions [1]–[5] (Table 5), this being necessary for statistical adequacy purposes.

This form of specification often demands (i) recasting assumptions about unobservable errors into equivalent assumptions in terms of the observable process $\{Y_t, k \in \mathbb{N}\}$, as well as (ii) unveiling hidden assumptions; see Spanos (1986). To illustrate this issue note that assumptions [1]–[5] (Table 5) imply that the error term $(u_t | Y_{t-1}^m)$ defines a [i] Normal (N), [ii] Martingale-difference (M-d) process:

$$(u_t, Y_{t-1}^m) \sim \text{NM-d}\left(0, \sigma_0^2\right), \quad t \in \mathbb{N},$$

but the converse is not true. The error assumptions in (44) do not provide a complete set of assumptions for the AR(p) model, because [i]–[ii] allow for $t$-varying coefficient parameters, in the sense that they hold even if $u_t = Y_t - \alpha_0(t) - \sum_{i=1}^{p} \alpha_i(t) Y_{t-i}$. Hence, in terms of (44), the crucial assumption [5] is veiled, and thus rarely tested in practice.

### 5.2. Statistical adequacy and model selection

In securing statistical adequacy the optimal value of $p$ is decided as part of the validation process for all [1]–[5] assumptions (Table 5), with particular emphasis placed on [4]; see Andreon and Spanos (2003). It can be shown that thorough M-S testing can guard against both underfitting and overfitting since both induce detectable systematic information in the residuals; see Spanos (1986).

This perspective questions the role of Akaike-type model selection procedures in determining $p$, which are widely used in econometrics (see Greene, 2003; Lutkepohl, 2005). As argued above, one of the reasons why the capacity of these procedures to ensure the reliability of inference is severely impaired is because statistical adequacy is ignored. What if one were to secure the statistical adequacy of the prespecified family of models $\{M_0(z), i = 1, 2, \ldots, m\}$ first, and then apply these model selection procedures? The fact of the matter is that establishing statistical adequacy renders these model selection procedures redundant. But even if we ignore that, model selection is likely to yield unreliable inference because, as argued above, their selection is tantamount to applying N–P testing with unknown error probabilities. Indeed, the fact that BIC selected the correct model in Table 4 was largely coincidental.

### Example 3 (Continued). To illustrate the unreliability of Akaike-type model selection procedures further, let $y_t$, the US population annual data (1955–1989), and consider the selection of a model within the AR(k, p) family:

$$AR(k, p) : y_t = \delta_0 + \sum_{i=1}^{k} \delta_i t + \sum_{i=1}^{p} a_i Y_{t-i} + \epsilon_t.$$  

(45)

In total accord with (31), AR(1, 2) is chosen on statistical adequacy grounds:

$$Y_t = 12.279 + .153 t + 1.560 Y_{t-1} - .628 Y_{t-2} + \hat{\epsilon}_t,$$  

(3.507) \quad (0.47) \quad (1.13) \quad (1.13)

$R^2 = .999$, $s = .1477$.  

(46)

In contrast, all three procedures, AIC, BIC and HQIC (see Table 6), selected the AR(3, 2) model. By comparing the results of Tables 4 and 6, it becomes clear that dropping the insignificant terms $(x_t, x_{t-1}, x_{t-2})$ from (31) to render the model more parsimonious, resulted in the BIC switching its selection to the wrong model!

### 5.3. Mis-specification (M-S) testing and respecification

The question that one might naturally pose at this stage is that, despite the apparent differences sketched above, both model selection and the model specification procedures come down to comparing one statistical model to another to find out which one is more appropriate. Such a view represents a misleading oversimplification.

A closer look at the above specification argument for AR(p) reveals that one is not choosing a statistical model as such, but a probabilistic structure for the stochastic process $\{Y_t, k \in \mathbb{N}\}$ that would render data $y_0$ a typical realization thereof; $M_0(y)$ constitutes a particular parameterization of this structure. This standpoint sheds very different light on the problem of underdetermination in this context. There can be two statistically adequate models only when they represent two alternative parameterizations of the same probabilistic structure; see Spanos (2007a). The choice between them is made using other criteria, including the substantive questions of interest.

The selected model $M_0(y)$ is viewed as an element of the set $\mathcal{P}(y)$ which includes all possible statistical models that could have given rise to data $y_0$. But how does one narrow down a possibly infinite set $\mathcal{P}(y)$ to one model $M_0(y)$? The narrowing down is attained by partitioning $\mathcal{P}(y)$ using probabilistic assumptions from three broad categories: Distribution (D), Dependence (M) and Heterogeneity (H); see Spanos (1995).
Example 6. The partitioning by reduction is illustrated in Fig. 5 in the case of the simple Normal model which is based on the reduction assumptions that \{Y_k, k ∈ N\} is (D) Normal, (M) Independent and (H) Identically Distributed, denoted by \(Y_k \sim \text{NID}(\mu, \sigma^2), k \in \mathbb{N}\), a model that seems appropriate for the data in Fig. 1.

The tripartite partitioning also plays a crucial role in M-S testing based on
\[ H : f^*(y) ∈ \mathcal{M}_0(y) \quad \text{vs.} \quad \overline{H} : f^*(y) \in \{\mathcal{P}(y) - \mathcal{M}_0(y)\}, \] (47)
where \(f^*(y)\) denotes the ‘true’ distribution of the sample. The probing beyond the boundaries of \(\mathcal{M}_0(y)\) raises several conceptual and technical issues concerning its effectiveness and reliability. The partitioning of \(\mathcal{P}(y)\) creates a framework wherein one can formally assess the model assumptions, relating to \(\{Y_k, k \in \mathbb{N}\}\), using informed M-S testing, because it provides an exhaustively complete probing strategy. Changing the original reduction assumptions in deliberative ways, in light of the statistical systematic information distilled by a statistical model for primary inferences. When modeling with observational data, the estimated \(\mathcal{M}_0(y)\) is often both statistically and substantively inadequate, and one has no way to delineate the two; is the theory wrong or are the (implicit) inductive premises invalid for data \(\mathcal{Z}_0\)? To assert this impenetrable dilemma, error statistics proposes to distinguish, ab initio, between statistical and substantive information and then bridge the gap between them by a sequence of interconnecting models which enable one to delineate and probe for the potential errors at different stages of modeling. From the theory side, the substantive information is initially encapsulated by a theory model and then modified into a structural one \(\mathcal{M}_0(y)\) to render it estimable with data \(\mathcal{Z}_0\). From the data side, the statistical information is distilled by a statistical model \(\mathcal{M}_0(y)\) whose parameterization is chosen with a view to render \(\mathcal{M}_0(y)\) a reparameterization/restriction thereof. The statistical adequacy of \(\mathcal{M}_0(y)\) is secured first in order to ensure the reliability of the procedures for appraising substantive claims.

5.4. The error statistical approach: Taking stock

Returning to the methodological questions (I)–(V) raised in the introduction, one can summarize the answers proposed by the error statistical approach as follows.

(I) The set of all possible models \(\mathcal{P}(z)\) can be narrowed down to a single model \(\mathcal{M}_0(z)\) using a three-way partitioning based on probabilistic assumptions pertaining to the process \(\{Z_k, k \in \mathbb{N}\}\) underlying data \(\mathcal{Z}_0\). By definition \(\mathcal{P}(z)\) includes a true probabilistic structure for \(\{\mathcal{Z}_0, k \in \mathbb{N}\}\) – one that would render data \(\mathcal{Z}_0\) a truly typical realization thereof – but there is no guarantee that one will always be able to parameterize this structure to get an operational model, whatever the chance regularities in data \(\mathcal{Z}_0\). That will depend on whether this true probabilistic structure has sufficient invariant features to be adequately parameterized by constant parameters \(\theta\).

(II) The adequacy of such a statistical model \(\mathcal{M}_0(z)\) is assessed post hoc by probing its probabilistic assumptions (e.g., [1]–[5] in Tables 2 and 5) vis-à-vis data \(\mathcal{Z}_0\) using thorough M-S testing to secure their validity. Statistical adequacy answers the question ‘when does \(\mathcal{M}_0(z)\) account for the chance regularities in data \(\mathcal{Z}_0\)?’

(III) Foisting the substantive information on the data by estimating the structural model \(\mathcal{M}_0(z)\) directly is invariably a rash strategy because statistical specification errors are likely to undermine the prospect of reliably evaluating the relevant errors for primary inferences. When modeling with observational data, the estimated \(\mathcal{M}_0(z)\) is often both statistically and substantively inadequate, and one has no way to delineate the two; is the theory wrong or are the (implicit) inductive premises invalid for data \(\mathcal{Z}_0\)? To assert this impenetrable dilemma, error statistics proposes to distinguish, ab initio, between statistical and substantive information and then bridge the gap between them by a sequence of interconnecting models which enable one to delineate and probe for the potential errors at different stages of modeling. From the theory side, the substantive information is initially encapsulated by a theory model and then modified into a structural one \(\mathcal{M}_0(z)\) to render it estimable with data \(\mathcal{Z}_0\). From the data side, the statistical information is distilled by a statistical model \(\mathcal{M}_0(z)\) whose parameterization is chosen with a view to render \(\mathcal{M}_0(z)\) a reparameterization/restriction thereof. The statistical adequacy of \(\mathcal{M}_0(z)\) is secured first in order to ensure the reliability of the procedures for appraising substantive claims.

(IV) Error statistics proposes a blending in of the Fisherian and Neyman–Pearson (N–P) perspectives that weaves a coherent frequentist inductive reasoning anchored firmly on error probabilities. The key is provided by realizing that the p-value is a post-data error probability and type I and type II are pre-data error probabilities, and that they fulfill crucial complementary roles. Pre-data error probabilities are used to appraise the generic capacity of inference procedures, and post-data error probabilities are used to bridge the

Table 6
AIC, BIC and HQIC based on (45).

<table>
<thead>
<tr>
<th>Model</th>
<th>AIC0 = ln((\hat{\theta})) + 2K/n0</th>
<th>Rank</th>
<th>BIC0 = 2K/n0</th>
<th>Rank</th>
<th>HQIC0 = 3K/n0</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR(1, 1)</td>
<td>ln(0.38545) + [2(4)/35] = −3.027</td>
<td>8</td>
<td>−2.850</td>
<td>7</td>
<td>−2.966</td>
<td>7</td>
</tr>
<tr>
<td>AR(1, 2)</td>
<td>ln(0.19320) + [2(5)/35] = −3.661</td>
<td>3</td>
<td>−3.439</td>
<td>2</td>
<td>−3.584</td>
<td>3</td>
</tr>
<tr>
<td>AR(1, 3)</td>
<td>ln(0.18818) + [2(6)/35] = −3.630</td>
<td>4</td>
<td>−3.363</td>
<td>3</td>
<td>−3.538</td>
<td>4</td>
</tr>
<tr>
<td>AR(2, 1)</td>
<td>ln(0.03640) + [2(5)/35] = −3.029</td>
<td>7</td>
<td>−2.807</td>
<td>8</td>
<td>−2.952</td>
<td>8</td>
</tr>
<tr>
<td>AR(2, 2)</td>
<td>ln(0.19150) + [2(6)/35] = −3.613</td>
<td>5</td>
<td>−3.346</td>
<td>5</td>
<td>−3.521</td>
<td>5</td>
</tr>
<tr>
<td>AR(2, 3)</td>
<td>ln(0.18795) + [2(7)/35] = −3.574</td>
<td>6</td>
<td>−3.263</td>
<td>6</td>
<td>−3.467</td>
<td>6</td>
</tr>
<tr>
<td>AR(3, 1)</td>
<td>ln(0.035610) + [2(6)/35] = −2.992</td>
<td>9</td>
<td>−2.726</td>
<td>9</td>
<td>−2.900</td>
<td>9</td>
</tr>
<tr>
<td>AR(3, 2)</td>
<td>ln(0.15525) + [2(7)/35] = −3.765</td>
<td>1</td>
<td>−3.454</td>
<td>1</td>
<td>−3.658</td>
<td>1</td>
</tr>
<tr>
<td>AR(3, 3)</td>
<td>ln(0.15525) + [2(8)/35] = −3.708</td>
<td>2</td>
<td>−3.353</td>
<td>4</td>
<td>−3.585</td>
<td>2</td>
</tr>
</tbody>
</table>
gap between the coarse ‘accept/reject’ and evidence provided by data \( z_0 \) for or against substantive claims; see Cox and Mayo (2010).

(V)’ Post-data error probabilities can be used to address both the fallacy of acceptance and the fallacy of rejection, using a post-data evaluation of inference based on severe testing reasoning. This amounts to establishing the smallest (largest) discrepancy \( \gamma \geq 0 \) from \( H_0 \) warranted by data \( z_0 \), associated with the N–P decision to accept (reject) \( H_0 \); see Mayo and Spanos (2006) and Mayo and Cox (2006).

6. Methodological issues raised by M-S testing

As argued above, statistical adequacy renders the relevant error probabilities ascertainable by ensuring that the nominal error probabilities are approximately equal to the actual ones. Spanos and McGuirk (2001) demonstrated that even seemingly minor departures from the assumptions of \( \mathcal{M}_0(z) \) can have devastating effects on the reliability of inference; see also Spanos (2009). In light of these, why is there such unwillingness to secure statistical adequacy using M-S testing in applied econometrics?

One possible explanation is that M-S testing is invariably viewed as defendable against several methodological charges including double-use of data, infinite regress, circularity and pre-test bias; see Kennedy (2008). Let us revisit these issues.

6.1. M-S testing and double-use of data

In the context of the error statistical approach it is certainly true that the same data \( z_0 \) are being used for two different purposes: (a) to test primary hypotheses in terms of the unknown parameter(s) \( \theta \), and (b) to assess the validity of the prespecified model \( \mathcal{M}_0(z) \), but ‘does that constitute an illegitimate double-use of data?’ The short answer is no, because, first, (a) and (b) pose very different questions to data \( z_0 \), and second, the probing takes place within versus outside \( \mathcal{M}_0(z) \), respectively.

Neyman–Pearson testing assumes that \( \mathcal{M}_0(z) \) is adequate, and poses questions within its boundaries. In contrast, the question posed by M-S testing is whether or not the particular data \( z_0 \) constitute a ‘truly typical realization’ of the stochastic mechanism described by \( \mathcal{M}_0(z) \), and the probing takes place outside its boundaries, i.e. in \( [P(z) - \mathcal{M}_0(z)] \); see Spanos (2000). Indeed, one can go as far as to argue that the answers to the questions posed in (a) and (b) rely on distinct information in \( z_0 \).

Spanos (2007b) showed that, for many statistical models, including the simple Normal and the Normal/linear regression (Table 2) models, M-S testing can be based solely on a maximal ancillary statistic \( \mathcal{R}(Z) := (R_1, \ldots, R_{n-m}) \), which is independent of a complete sufficient statistic \( S(Z) := (S_1, \ldots, S_m) \) used solely for primary inferences. This is the case when evaluating the distribution of the sample \( f(z; \theta) \) simplifies as follows:

\[
f(z; \theta) = |J| \cdot f(s; \theta) \cdot f(r),
\]

\[
\mathbb{V}(s, r) \in \mathbb{R}_+^m \times \mathbb{R}_+^{m-n}, \tag{48}
\]

where \(|J|\) denotes the Jacobian of the transformation \( Z \rightarrow (S(Z), \mathcal{R}(Z)) \). This means that all primary inferences can be based exclusively on \( f(s; \theta) \), and \( f(r) \) (free of \( \theta \)) can be used to appraise the validity of the statistical model in question.

Example 4 (Continued). For \( Z_k \sim \mathcal{N}IID(\mu, \sigma^2) \), \( k = 1, \ldots, n \), the minimal sufficient statistic is \( S := (\bar{Z}_n, s^2) \) and the maximal ancillary statistics is \( \mathcal{R}(Z) = (\bar{z}_2, \ldots, \bar{z}_n) \), where \( \bar{z}_k = (\sqrt{n}(Z_k - \bar{Z}_n))/s \), \( k = 1, 2, \ldots, n \), are known as the Studentized residuals.

This view calls into question the argument by Claeskins and Hjort (2008), p. xi:

“Uncertainties involved in the first step [specification] must be taken into account when assessing distributions, confidence intervals, etc. That such themes have been largely underplayed in theoretical and practical statistics has been called ‘the quiet scandal of statistics.’ … Model averaging can help to develop methods for better assessment and better construction of confidence intervals, p-values, etc.”

As argued above, there is nothing scandalous about separating the two steps: it is a matter of judicious modeling. Allowing specification errors to vitiate inferential error probabilities will derail any learning from data about the underlying mechanism, and no amount of averaging over mis-specified models can redeem the reliability of inference. Even in the best case scenario where \( \mathcal{M}_0(z) = \lambda \mathcal{M}_0(z) + (1 - \lambda) \mathcal{M}_0(z), 0 < \lambda < 1 \), \( \mathcal{M}_0(z) \) is statistically adequate but \( \mathcal{M}_0(z) \) is mis-specified, the result of averaging is to ruin the reliability of inference based on \( \mathcal{M}_0(z) ; \mathcal{M}_0(z) \) is a mis-specified model. Learning from data can only occur when the inferential error probabilities relate directly to an adequate description of the underlying mechanism, and not when the assumed model \( \mathcal{M}_0 \) includes \( \mathcal{M}_0(z) \), which could not have contributed in generating data \( z_0 \). This is not to deny that model averaging can play a role in the context of substantive adequacy, when one is dealing with statistically adequate models based on different data\( \{ \mathcal{M}_0(z), i = 1, \ldots, m \} \), e.g. regressions with different regressors.

6.2. M-S testing and infinite regress/circularity charges

The infinite regress charge is often articulated by claiming that each M-S test relies on a set of assumptions, and thus it assesses the assumptions of the model \( \mathcal{M}_0(z) \) by invoking the validity of its own assumptions, trading one set of assumptions with another ad infinitum. Indeed, this reasoning is often circular because some M-S tests inadvertently assume the validity of the very assumption being tested!

A closer look at the reasoning underlying M-S testing reveals that both charges are misplaced. First, the scenario used in evaluating the type I error invokes no assumptions beyond those of \( \mathcal{M}_0(z) \), since every M-S test is evaluated under \( H_0 : \) all the probabilistic assumptions of \( \mathcal{M}_0(z) \) are valid.

Example 7. The runs test, using the residuals from an AR(p) model \( \{z_t, t = 1, 2, \ldots, n \} \), is an example of an omnibus M-S test for assumptions \( 4, 5 \) (Table 5) based a test statistic: \( Z_0(Y) = R - E(R)/\sqrt{\text{Var}(R)} \), see Spanos (1999). For \( n \geq 40 \), the type I error probability evaluation is based on

\[
Z_0(Y) = \frac{R - ((2n - 1)/3)}{\sqrt{16n - 29}/90} \sim N(0, 1).
\]

Second, the type II error (and power), for any M-S test, is determined by evaluating the test statistic under certain forms of departures from the assumptions being appraised [no circularity], but retaining the rest of the model assumptions, or by choosing M-S tests which are insensitive to departures from the retained assumptions.

For the runs test, the evaluation under the alternative takes the form

\[
Z_0(Y) \sim N(\delta, \tau^2), \quad \delta \neq 0, \tau^2 > 0,
\]

where \([4] \) and \([5] \) denote specific departures from these assumptions considered by the test in question; note that the runs test is insensitive to departures from Normality. The type of departures implicitly or explicitly considered by the M-S test in question will
affect the power of the test in a variety of ways, and one needs to apply a battery of different M-S tests to ensure broad probing capacity and self-correcting in the sense that the effect of any departures from the maintained assumptions is also detected.

In practice, potential problems such as circular reasoning, inadequate probing and erroneous diagnoses can be circumvented by employing the following.

(a) Judicious combinations of parametric, nonparametric, omnibus and simulation-based tests, probing as broadly as possible and invoking dissimilar assumptions.

(b) Astute ordering of M-S tests so as to exploit the interrelationship among the model assumptions with a view to 'correct' each other's diagnosis.

(c) Joint M-S tests (testing several assumptions simultaneously) designed to avoid 'erroneous' diagnoses as well as minimize the maintained assumptions.

These strategies enable one to argue with severity that, when no departures from the model assumptions are detected, the validated model provides a reliable basis for appraising substantive claims; see Spanos (2000) and Mayo and Spanos (2004).

6.3. M-S testing/respecification and pre-test bias

The question sometimes raised is whether the above error statistical strategies of M-S testing and respecification are vulnerable to the charge of pre-test bias. To discuss the merits of this charge, consider the Durbin–Watson test for assessing the assumption of no autocorrelation for the linear regression errors, based on (see Greene, 2003):

\[ H_0 : \rho = 0 \text{ vs. } H_1 : \rho \neq 0. \]

Step 1. The pre-test bias perspective interprets this M-S test as equivalent to choosing between the following two models:

\[ M_0(x) : y_t = \beta_0 + \beta_1 x_t + u_t, \]
\[ M_1(z) : y_t = \beta_0 + \beta_1 x_t + u_t + \rho u_{t-1} + \varepsilon_t. \]  

\[ \text{(49)} \]

Step 2. This is then formalized into a choice between two estimators of \( \beta_1 \) in decision-theoretic terms using the pre-test estimator:

\[ \hat{\beta}_1 = (1 - \lambda) \tilde{\beta}_1 + \lambda \tilde{\beta}_1, \quad \text{where } \lambda = \begin{cases} 1, & \text{if } H_0 \text{ is accepted} \smallskip \ 0, & \text{if } H_0 \text{ is rejected} \end{cases}; \]

\[ \text{(50)} \]

\( \tilde{\beta}_1 \) is the OLS estimator under \( H_0 \), and \( \tilde{\beta}_1 \) is the GLS estimator under \( H_1 \).

Step 3. This perspective claims that the relevant error probabilities revolve around the Mean Square Error (MSE) of \( \tilde{\beta}_1 \), whose sampling distribution is usually non-Normal, biased and has a highly complicated variance; see Leeb and Pflüschter (2005).

When viewed in the context of the error-statistical approach, the pre-test bias argument, based on (50), seems highly questionable on several grounds.

First, it misconstrues M-S testing by recasting it as a decision-theoretic estimation problem based on a loss function. As argued discerningly by Hacking (1965), pp. 31:

“Deciding that something is the case differs from deciding to do something.”

M-S testing poses the canonical question whether \( M_0(z) \) is statistically adequate, i.e. it accounts for the chance regularities in data \( x_0 \) or not; it is not concerned with selecting one of two models come what may. Having said that, one can potentially construct an M-S test with a view to assess a subset of the model assumptions by viewing an alternative model \( M_1(z) \) as a result of narrowing \([\theta(z) - M_0(z)] \) down to a single alternative model (parametrically) encompasses \( M_0(z) \); see Spanos (1999).

As argued in Section 3.4, however, when the ultimate inference is concerned with whether \( M_0(z) \) is statistically adequate, the relevant errors are

(i) the selected model is inadequate but the other model is adequate, or
(ii) both models are inadequate.

In contrast, \( E(\hat{\beta}_1 - \beta_1)^2 \) evaluates the expected loss resulting from the modeler's supposedly tacit intention to use \( \hat{\beta}_1 \) as an estimator of \( \beta_1 \). Is there a connection between \( E(\hat{\beta}_1 - \beta_1)^2 \), for all \( \beta_1 \in \mathbb{R} \), and the errors (i)-(ii)? The short answer is no. The former evaluates the expected loss stemming from one's (misguided) intentions, but the latter pertains to the relevant error probabilities (type I and type II) associated with the inference that one of the two models is statistically adequate; Spanos (forthcoming).

Second, when an M-S test supposedly selects the alternative \( (M_0(z)) \) — the implicit inference being that it is statistically adequate — the pre-test argument perpetrates the fallacy of rejection [evidence against \( H_0 \) is misinterpreted as evidence for a particular \( H_1 \)]. The validity of \( M_0(z) \) needs to be established separately by testing its own assumptions. Hence, in an M-S test one should never accept the alternative without further testing: Spanos (2000).

Third, when an M-S test supposedly selects the null \( (M_0(z)) \) — the implicit inference being that it is statistically adequate — the inference is problematic for two reasons. This inference is problematic for two reasons. First, given the multitude of assumptions constituting a model, there is no single M-S test based on a parametrically encompassing model \( M_0(z) \), that could, by itself, establish the statistical adequacy of \( M_0(z) \). Second, the inference is vulnerable to the fallacy of acceptance [no evidence against \( H_0 \) is misinterpreted as evidence for it]. It is possible that the particular M-S test did not reject \( M_0(z) \) because it had very low power to detect an existing departure. In practice this can be remedied using additional M-S tests with higher power to cross-check the results, or/and use a post-data evaluation of inference to establish the warranted discrepancies from \( H_0 \).

In summary, instead of devising ways to circumvent the fallacies of rejection and acceptance to avoid erroneous inferences, the pre-test bias argument embraces them by recasting the original problem (in step 1), formalizes them (in step 2), and evaluates risks (in step 3) that have no bearing on erroneously inferring that the selected model is statistically adequate. The pre-test bias charge is ill-conceived because it misrepresents model validation as a choice between two models come what may.

7. Summary and conclusions

Akaike-type model selection procedures, relying on trading goodness-of-fit against parsimony, often give rise to unreliable inferences primarily because they (a) assume away the problem of statistical model validation, and (b) ignore the relevant error probabilities for the inferences reached. The same problems arise when one relies on mathematical approximation theory (curve fitting) to select a model within a presupposed (parametric or nonparametric) family.

The paper called for a return to the problem of statistical model specification as originally envisaged by Fisher (1922) with two crucial differences. The first pertains to having a purely probabilistic construal of the notion of a statistical model, and the second calls for addressing the specification problem in the context of a refinement/extension of the original frequentist framework known as error statistics, which emphasizes the probing of the different ways an inference might be incorrect. It was shown that the issues (a)-(b), as well as the foundational problems (I)-(IV) raised in Section 1, can be addressed in the context of this framework.
Using statistical adequacy as the sole criterion for assessing when a statistical model \( M_k(z) \) ‘accounts for the chance regularities in data \( z_0 \)’ renders the relevant error probabilities ascertainable and can obviate statistical mis-specification, thus securing the reliability of inference. The key is provided by viewing \( M_k(z) \) as a parameterization of the probabilistic structure of the process \( \{ Z_k, k \in \mathbb{N} \} \) underlying data \( z_0 \), rendering its assumptions testable and nesting parametrically the structural model \( M_0(z) \). Disentangling the specification and inference facets of inductive inference is particularly important to prevent statistical specification errors from vitiating the inferential error probabilities and thus forestall any learning from data. This is because foisting the substantive information on the data, by estimating the structural model \( M_0(z) \) directly, yields estimated models which are often both statistically and substantively inadequate, but one has no way to adjudicate whether the theory is wrong or the inductive premises are invalid for data \( z_0 \). Judicial M-S testing and informed respecification – guided by graphical techniques – can secure the statistical adequacy of \( M_k(z) \), which can be subsequently used as a basis for reliable inductive inference in probing substantive questions of interest. The various charges leveled against M-S testing and informal respecification, including double-use of data, circularity, infinite regress and pre-test bias, stem from an inadequate appreciation of the underlying methodological issues. When viewed in the context of the error statistical framework, these charges are shown to be ill-conceived and misdirected.

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Appendix. Mathematical approximation theory: A brief summary of relevant results

Mathematically the most effective way to render the curve-fitting problem tractable is to view it in the context of a normed linear space, say \( \mathcal{L} \), where the ‘true’ function \( h \in \mathcal{L} \), and the approximating function \( g_m(\cdot) \) belongs to a subset \( \mathcal{g} \subset \mathcal{L} \). The notion of ‘best’ is defined in terms of a norm \( \| \cdot \| : \mathcal{L} \rightarrow [0, \infty) \), and satisfies certain properties generalizing the notion of length; see Luenberger (1969).

Example 8. Let \( C[a, b] \) consist of all real-valued continuous functions on the real interval \( [a, b] \subset \mathbb{R} \), together with the \( \infty \)-norm:

\[
\| f \|_\infty = \max_{a \leq x \leq b} |f|.
\]

(51)

It can be shown that \( (C[a, b], \| \cdot \|_\infty) \) defines a normed linear space. Other norms in the context of \( C[a, b] \) are special cases of

\[
\| f \|_p = \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad p \geq 1.
\]

(52)

The norm defines a distance between any two elements \( (f, g) \) in \( \mathcal{L} \) via the metric

\[
d(f, g) = \| f - g \| \quad \text{for any } (f, g) \in \mathcal{L},
\]

where \( d(\cdot, \cdot) : (\mathcal{L} \times \mathcal{L}) \rightarrow [0, \infty) \), and satisfies certain properties. The pair \( (\mathcal{L}, d(\cdot, \cdot)) \) defines a metric space induced by \( (\mathcal{L}, \| \cdot \|) \). Similarly, one can define the notion of an inner (scalar) product via

\[
\sqrt{\langle f, f \rangle} = \| f \|, \quad \text{for any } f \in \mathcal{L},
\]

inducing an inner product space \( (\mathcal{L}, \langle \cdot, \cdot \rangle) \), where \( d(f, g) = \langle f - g, f - g \rangle^{\frac{1}{2}} \); see Powell (1981). The additional mathematical structure induced by an inner product (to define angles, and thus orthogonality via \( \langle f, g \rangle = 0 \)) is needed when the problem requires one to go beyond existence and uniqueness results to construct the approximating function \( g_m(\cdot) \) explicitly.

Example 9. For the normed linear space \( (C[a, b], \| \cdot \|_\infty) \), the induced metric space is \( (C[a, b], d_\infty(\cdot, \cdot)) \), where

\[
d_\infty(f, g) = \max_{a \leq x \leq b} |f(x) - g(x)|.
\]

In a normed linear space one can pose three interrelated questions of interest:

(i) Does there exist in \( \mathcal{g} \) a best approximation \( g^* (x) \) of \( f(x) \) ?

(ii) When \( g^*(x) \) exists, is it unique?

(iii) How can one construct \( g^*(x) \)?

(iv) How adequate is the approximation rendered by the constructed \( g^*(x) \) ?

Let us consider each of these issues briefly.

The existence result is easy to ensure when \( \mathcal{g} \) is chosen to be a compact subset of a normed linear space \( \mathcal{L} \). The cornerstone of such existence results is a famous theorem whose general form is as follows.

**Theorem 1.** Let \( f(x) \) be an upper semicontinuous functional on \( \mathcal{g} \), a compact subset of a normed linear space \( (\mathcal{L}, \| \cdot \|) \); then \( f(x) \) achieves a maximum on \( \mathcal{g} \); see Luenberger (1969). A special case of this theorem is known as Weierstrass’ theorem, which ensures that, when \( h(x) \) is in \( C[a, b], \| \cdot \|_\infty \) for any \( \epsilon > 0 \) there exists an integer \( N(\epsilon) \) such that, for \( m > N(\epsilon) \),

\[
|h(x) - g_m(x; \alpha_m)| < \epsilon, \quad \text{for all } x \in [a, b],
\]

(53)

where \( g_m(x; \alpha_m) = \sum_{i=0}^{m} \alpha_i x^i, m \geq 1 \). The norm underlying (53) is the \( \infty \)-norm in (51), giving rise to the metric

\[
d_\infty(h(x), g_m(x; \alpha_m)) = \max_{a \leq x \leq b} |h(x) - g_m(x; \alpha_m)|,
\]

(54)

and the mode of convergence is known as uniform. Note that the coefficients \( \alpha_m := (\alpha_0, \alpha_1, \ldots, \alpha_m) \) depend crucially on \( m \), and as the degree of the polynomial increases these coefficients change with it.

The uniqueness result depends crucially on the convexity of both \( \mathcal{g} \) and \( \| \cdot \| \).

**Theorem 2.** Let \( \mathcal{g} \) be a convex set in a normed linear space \( (\mathcal{L}, \| \cdot \|) \), whose norm is strictly convex. Then, for all \( f \in \mathcal{L} \), there is a unique best approximation in \( \mathcal{g} \).

Example 8 (Continued). In the case of \( C[a, b] \), when \( \mathcal{g} \) is a finite-dimensional linear subspace (convex set), the 2-norm

\[
\| f \|_2 = \left( \int_a^b |f(x)|^2 \, dx \right)^{\frac{1}{2}},
\]

(55)

ensures uniqueness because \( \| f \|_2 \) is strictly convex. However, the \( \infty \)-norm in (51) or the 1-norm

\[
\| f \|_1 = \left( \int_a^b |f(x)| \, dx \right)
\]

(56)

is not strictly convex. Note that in general (Powell, 1981):

\[
\| f \|_1 \leq (b - a)^{\frac{1}{2}} \| f \|_2 \leq (b - a) \| f \|_\infty \quad \text{for all } f \in C[a, b].
\]

(57)

This means that in the case of the normed linear spaces \( (C[a, b], \| \cdot \|_1) \) and \( (C[a, b], \| \cdot \|_\infty) \), one needs to impose further
restrictions on $\mathcal{g}$ or $h(x)$ to ensure uniqueness. One such restriction on a linear subspace $\mathcal{g}$ of $C[a, b]$, $\dim(\mathcal{g}) = m + 1$, is the Haar condition.

Haar condition. For any $\phi(x) \in \mathcal{g}$ that is not the zero element, the number of roots of the equation $[\phi(x) = 0, x \in [a, b]]$ is at most $m$.

In the case where $\mathcal{g}$ is a set of polynomials of the form $g_m(x; \alpha) = \alpha_0 x^m + \cdots + \alpha_m x^0$, this condition is satisfied if $\dim(\mathcal{g}) = m + 1$ and $g_m(x; \alpha)$ can have at most $m$ distinct zeros.

Chebyshev set. A closely related condition is to exchange the base functions $\{1, x, x^2, \ldots, x^m\}$ with a Chebyshev set of generalized polynomials: $\{\phi_i(x), i = 0, 1, \ldots, m\}$, defined on $[a, b]$, such that every non-trivial linear combination $g_m(x; \alpha) = \alpha_0 x^m + \cdots + \alpha_m x^0$ has at most $m$ distinct zeros on $[a, b]$; $\alpha_0 = \cdots = \alpha_m = 0$ is the trivial case; see Powell (1981).

 Necessary and sufficient conditions. In the context of the normed linear space $(C[a, b], \|\cdot\|_\infty)$, where a Chebyshev set $\{\phi_i(x), i = 0, 1, \ldots, m\}$ spans $\mathcal{g}$, the Haar condition ensures that the best approximating polynomial $g_m(x; \alpha^*)$ is unique if and only if the error function $e(x; m) = h(x) - g_m(x; \alpha^*)$, $x \in [a, b]$, changes sign more than $m + 1$ times as $x$ varies over $[a, b]$; see Cheney (1982). It is important to emphasize that this result does not say that points $\{x_k, k = 0, 1, \ldots, m\}$, where the successive change of sign in the residuals $\{e(x_k); m\} = 0, 1, \ldots, m$ alternates in sign at least $m + 2$ times.

Piecewise approximation. Of particular interest in mathematical approximation theory are the results pertaining to local (piecewise) approximation using splines, where the interval $[a, b]$ is partitioned at $N + 2$ knots $a = x_0 < x_1 < x_2 < \cdots < x_N < x_{N+1} = b$, and a polynomial of degree $m$, say $g_m(x)$, is fitted over each interval $[x_k, x_{k+1}], k = 0, 1, \ldots, N$. The most widely used piecewise polynomials are splines of degree $m$ (often $m = 2$), which have $(m - 1)$ continuous derivatives at the knots. Analogous oscillation theorems are also applied to the case of local (piecewise) approximation using splines; see Watson (1980), pp. 157–171.

It turns out that in cases where the approximating function $g^*(x)$ is unique, it can be represented by a mapping from $L$ to $\mathcal{g}$, say $g^*(x) = P_h(h(x))$, and the structure of $P_h(.)$ is of value in considering the question of constructing such best approximations. This mapping is often a linear projection operator which is characterized by the property that $P_h(f(x))\| = P_h(f(x))$, for all $f \in L$.

A sufficient condition for $P(.)$ to be a projection is to satisfy the condition $P_h(g(x)) = g(x)$, for all $g \in \mathcal{g}$.

Theorem 4. Let $\mathcal{g}$ be a finite-dimensional linear space in a normed linear space $(L, \|\cdot\|)$, such that, for every $f \in L$, there is a unique best approximation in $\mathcal{g}$, say $P_h(f)$; then the operator $P_h$ is continuous. Moreover, when $P(h)$ satisfies condition (59), then for $d^* = \min_{f \in L} \|f - g\|$, the error of the approximation $P_h(f)$ satisfies the bound (Powell, 1981) $\|f - P_h(f)\| \leq [1 + \|P_h\|]d^*$.

This result is of interest in constructing $g^*(x) = P_h(f)$ because rounding errors can have substantial effects on the constructed approximations if $P_h$ is discontinuous. Viewing the approximating function as a projection also sheds additional light on both questions (iii) and (iv).

(iii) Constructing an approximating function often involves (a) the choice of the appropriate family of building block functions spanning $\mathcal{g}$ which, for reasons of existence and uniqueness, often take the form $g_m(x; \alpha) = \sum_{i=0}^{m} \alpha_i \phi_i(x)$, (60)

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)$, and $[\phi_i(x), i = 0, 1, \ldots, m]$ is a base (Chebyshev) set of generalized polynomial functions, and (b) an algorithm that chooses the best approximating function within this family; see Powell (1981).

(iv) Assessing the adequacy of the best approximating function involves constructing upper bounds for the approximating error or even deriving explicit forms of the error function under certain circumstances where additional smoothness conditions are imposed on $h(x)$. Typical results in this context are the following.

Theorem 5. Consider the problem of approximating $h(x)$, an element of $(C[a, b], \|\cdot\|_\infty)$, using $g_m(x; \alpha)$ in (60), and define the modulus of continuity of $h(x)$:

\[ \omega(\delta) = \sup_{|x - y| \leq \delta} |h(x) - h(y)|. \]

For $(x_1, x_2) \in [a, b], \delta > 0$.

The approximation error $\varepsilon(x; m)$ satisfies the Jackson-type upper bound:

\[ \varepsilon(x; m) \leq 6\omega(\delta) \frac{(b - a)}{2m}. \]

For more accurate bounds one needs to impose additional smoothness conditions on $h(x)$, such as the existence of derivatives up to order $k$. Such bounds are useful in appraising the behavior of $\varepsilon(x; m)$ as $m \to \infty$, and hence the appropriateness of the functions spanning $\mathcal{g}$; see Rivlin (1981) and Cheney (1982).

In the case of piecewise (local) approximation, such as splines, these Jackson-type upper bounds also depend on the length of the intervals $[x_k, x_{k+1}]$, say $h_k = [x_{k+1} - x_k], k = 0, 1, \ldots, N$. When one defines a uniform partition, i.e. $x_{k+1} - h_k$, the length $h$ is known as the smoothing parameter, which plays an important role in convergence results; see Watson (1980).

Theorem 6. Let $h(x) \in C[a, b]$. The Lagrange interpolation polynomial on a set of points $G_m(x) = \{x_k, k = 1, \ldots, n\}, n \geq m$, spanning the interval $[a, b]$, takes the form

\[ g_0(x; \alpha) = \sum_{i=0}^{m} \sum_{j=0}^{m} y_i \prod_{j=0}^{m} \left( x - x_j^* \right)^{\prod_{j=0}^{m} x_i^*}, \quad x \in [a, b], \]

where the interpolation points $(x_0^*, x_1^*, \ldots, x_n^*) \in [a, b]$ are chosen to be distinct. For a smooth enough function $h(x)$ (derivatives up to order $m + 1$ exist), the error is a systematic function of both $x$ and $m$, since

\[ \varepsilon(x, m) = \frac{d^{m+1}h(\xi)}{d^{m+1}x} \frac{1}{(m + 1)!} \prod_{j=0}^{m} (x - x_j^*), \quad \xi \in [a, b]. \]

(64) suggests that the error curve $\varepsilon(x, m)$ behaves like a polynomial in $x$, with $m + 1$ roots $(x_0^*, x_1^*, \ldots, x_m^*)$: $\varepsilon(x, m) = ax^{m+1} + bx^m + \cdots + b_{m-1}x + b_0$. Such an oscillating curve is also typical for error terms arising from the least squares approximation; see Watson (1980).
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